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THE TRANSFORMS OF PYTHAGOREAN AND QUADRATIC MEANS OF WEIGHTED SHIFTS

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Abstract. In this article, we introduce the transforms of Pythagorean and quadratic means of weighted shifts. We then explore how the transforms of weighted shifts behaves, in comparison with the Aluthge transform.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space and T be a bounded linear operator defined on $\mathcal{H}$ whose polar decomposition is $T = U|T|$. The *Aluthge transform* of T is the operator $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. This transform was first studied in [1] and has received much attention in recent years. One reason the Aluthge transform is interesting is in relation to the invariant subspace problem. We recall that the *Duggal transform* $\widetilde{T}^D = |T|U$ of T, which is first referred in [12]. Clearly, the spectrum of $\widetilde{T}$ $\left(\text{resp. } \widetilde{T}^D\right)$ equals that of T. For $\alpha \equiv \{\alpha_k\}_{k=0}^{\infty}$ a bounded sequence of positive real numbers (called *weights*), let $W_\alpha \equiv shift(\alpha_0, \alpha_1, \cdots) : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be the associated *unilateral weighted shift*, defined by $W_\alpha e_k := \alpha_k e_{k+1}$ (all $k \geq 0$), where $\{e_k\}_{n=0}^{\infty}$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$. For a shift W_α, we let $\widetilde{W_\alpha}$ be the Aluthge transform of W_α. Then we can see that $\widetilde{W_\alpha} = shift(\sqrt{\alpha_0\alpha_1}, \sqrt{\alpha_1\alpha_2}, \cdots) =: shift(\widetilde{\alpha}_0, \widetilde{\alpha}_1, \cdots)$ (called the shift of *the geometric mean* of a sequence). In [14] we study some properties of the mean transform $\widehat{T} := \frac{1}{2}(U|T| + |T|U) = \frac{1}{2}\left(U|T| + \widetilde{T}^D\right)$. Let $\widehat{W_\alpha}$ be the Mean transform of W_α. Then we have that $\widehat{W_\alpha} =$

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$shift\left(\frac{\alpha_0+\alpha_1}{2}, \frac{\alpha_1+\alpha_2}{2}, \cdots\right)$ =: $shift(\widehat{\alpha}_0, \widehat{\alpha}_1, \cdots)$ (called the shift of *the arithmetic mean* of a sequence). Thus, based on the arithmetic and geometric means of sequences just given above, it is natural to consider hamonic and quadratic means of sequences. For a weighted shift W_α, we let $\widetilde{W}_\alpha^H := shift\left(\frac{2\alpha_0\alpha_1}{\alpha_0+\alpha_1}, \frac{2\alpha_1\alpha_2}{\alpha_1+\alpha_2}, \cdots\right)$ be the *hamonic mean transform* of W_α and $\widetilde{W}_\alpha^Q := shift\left(\sqrt{\frac{\alpha_0^2+\alpha_1^2}{2}}, \sqrt{\frac{\alpha_1^2+\alpha_2^2}{2}}, \cdots\right)$ be the *quadratic mean transform* of W_α, respectively. We call the arithmetic, geometric and hamonic means *Pythagorean* means. In this article, we study some properties of the transforms of Pythagorean and quadratic means, and the following problems. From ([11, Theorem 2.8]), we knew that the Aluthge transform $\widetilde{B}_+$ of the Bergman shift $B_+ \equiv shift\left(\sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \cdots\right)$ is subnormal. Thus, we can ask:

Problem 1.1. *Is the hamonic mean transform $\widetilde{B_+}^H$ of B_+ subnormal?*

Problem 1.2. *Is the quadratic mean transform $\widetilde{B_+}^Q$ of B_+ subnormal?*

We can also ask:

Problem 1.3. *For $k \geq 1$, if W_α is k-hyponormal, does it follow that the transform $\widetilde{W}_\alpha^H$ $\left(\text{resp. } \widetilde{W}_\alpha^Q\right)$ k-hyponormal?*

Problem 1.4. *If W_α is subnormal with Berger measure μ, does it follow that $\widetilde{W}_\alpha^H$ $\left(\text{resp. } \widetilde{W}_\alpha^Q\right)$ subnormal? If it does, what is the Berger measure of $\widetilde{W}_\alpha^H$ $\left(\text{resp. } \widetilde{W}_\alpha^Q\right)$?*

Recall that $T \in \mathcal{B}(\mathcal{H})$ is *normal* if $T^*T = TT^*$, *subnormal* if $T = N|_{\mathcal{H}}$, where N is normal and $N(\mathcal{H}) \subseteq \mathcal{H}$, and *hyponormal* if $T^*T \geq TT^*$. It is called that $T \in \mathcal{B}(\mathcal{H})$ is *quasinormal* if T commutes with T^*T. It is well known that normal $\Longrightarrow$ quasinormal $\Longrightarrow$ subnormal $\Longrightarrow$ hyponormal.

2. Main results

For a weighted shift W_α, it is easy to see that W_α is hyponormal if and only if $\alpha_0 \leq \alpha_1 \leq \cdots$. Thus, the hamonic transform $\widetilde{W}_\alpha^H$ $\left(\text{resp. the quadratic transform } \widetilde{W}_\alpha^Q\right)$ of W_α is hyponormal if and only if $\frac{2\alpha_n\alpha_{n+1}}{\alpha_n+\alpha_{n+1}} \leq \frac{2\alpha_{n+1}\alpha_{n+2}}{\alpha_{n+1}+\alpha_{n+2}}$ $\left(\text{resp. } \sqrt{\frac{\alpha_n^2+\alpha_{n+1}^2}{2}} \leq \sqrt{\frac{\alpha_{n+1}^2+\alpha_{n+2}^2}{2}}\right)$ if and only if $\alpha_n \leq \alpha_{n+2}$ (all $n \geq 0$). Hence, if W_α is hyponormal, then the hamonic transform $\widetilde{W}_\alpha^H$ and the quadratic transform $\widetilde{W}_\alpha^Q$ are both hyponormal. The converse is not true in general. For example, let $W_\alpha \equiv shift\left(\frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, 1, \cdots\right)$. Then W_α is clearly not hyponormal but he hamonic transform $\widetilde{W}_\alpha^H = \frac{2}{3}U_+$ and the quadratic transform $\widetilde{W}_\alpha^Q = \sqrt{\frac{5}{8}}U_+$, so both transforms are subnormal.

We recall that for $k \geq 1$, T is *k-hyponormal* if

$$\begin{pmatrix} I & T^* & T^{*^2} & \cdots & T^{*^k} \\ T & T^*T & T^{*^2}T & \cdots & T^{*^k}T \\ T^2 & T^*T^2 & T^{*^2}T^2 & \cdots & T^{*^k}T^2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ T^k & T^{*^2}T^k & T^{*^2}T^k & \cdots & T^{*^k}T^k \end{pmatrix}_{(k+1)\times(k+1)} \geq 0.$$

The Bram-Halmos characterization of subnormality ([2, III.1.9]) can be paraphrased as follow: T is subnormal if and only if T is k-hyponormal for every $k \geq 1$ ([7, Proposition 1.9]). We recall that the *moments* of W_α are given as

$$\gamma_n \equiv \gamma_n(W_\alpha) := \begin{cases} 1, & \text{if } n = 0 \\ \alpha_0^2 \cdot \ldots \cdot \alpha_{n-1}^2, & \text{if } n > 0 \end{cases}. \tag{2.1}$$

We now consider whether the mean transform $\widehat{W}_\alpha$ of W_α preserves the k-hyponormality. We show that there exists a subnormal weighted shift W_α such that $\widehat{W}_\alpha$ is not 2-hyponormal. For this, we need the follwing lemmas.

LEMMA 2.1. ([3]) *Let $W_\alpha e_i = \alpha_i e_{i+1}$ $(i \geq 0)$ be a hyponormal weighted shift, and let $k \geq 1$. The following statements are equivalent:*

(i) *W_α is k-hyponormal.*

(ii) *The matrix*

$$(([W_\alpha^{*j}, W_\alpha^i]e_{n+j}, e_{n+i}))_{i,j=1}^k$$

is positive semi-definite for all $n \geq -1$.

(iii) *The matrix*

$$(\gamma_n\gamma_{n+i+j} - \gamma_{n+i}\gamma_{n+j})_{i,j=1}^k$$

is positive semi-definite for all $n \geq 0$, where as usual $\gamma_0 = 1$, $\gamma_n = \alpha_0^2 \cdots \alpha_{n-1}^2$ $(n \geq 1)$.

(iv) *The Hankel matrix*

$$H(k;n) := (\gamma_{n+i+j-2})_{i,j=1}^{k+1}$$

is positive semi-definite for all $n \geq 0$.

Lemma 2.2. (cf.[17]) *Let $M \equiv \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ be a 2×2 operator matrix, where A and C are square matrices and B is a rectangular matrix. Then*

$$M \geq 0 \iff \text{there exists } W \text{ such that } \begin{cases} A \geq 0 \\ B = AW \\ C \geq W^* AW. \end{cases}$$

For matrices $A, B \in M_n(\mathbb{C})$, we let $A \circ B$ denote their *Schur product*, i.e., $(A \circ B)_{ij} := A_{ij} B_{ij}$ $(1 \leq i, j \leq n)$. The following result is well known: If $A \geq 0$ and $B \geq 0$, then $A \circ B \geq 0$ ([15]). For matrices $A, B \in M_n(\mathbb{C})$, we let $A \circ B$ denote their *Schur product.* For $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$ and $\beta \equiv \{\beta_n\}_{n=0}^{\infty}$, the Schur product of α and β is defined by $\alpha \circ \beta := \{\alpha_n \beta_n\}_{n=0}^{\infty}$. Thus, for given two 1-variable subnormal weighted shifts W_α and W_β, their Schur product $W_\alpha \circ W_\beta$, which we denote by $W_{\alpha\beta}$, is subnormal. That is, if W_α and W_β are k-hyponormal $(k \geq 1)$ 1-variable weighted shifts, then the Schur product

(2.2) $W_{\alpha\beta} \equiv W_\alpha \circ W_\beta$ is a k-hyponormal 1-variable weighted shift [8].

Let $a, b, c, d \geq 0$ satisfy $ad - bc > 0$. Let $S(a, b, c, d) := shift(\alpha_0, \alpha_1, \alpha_2, \cdots)$, where $\alpha_n := \sqrt{\frac{an+b}{cn+d}}$ $(n \geq 0)$. Then we have:

Lemma 2.3. ([9]) *Let $a, b, c, d \geq 0$ satisfy $ad - bc > 0$. Then $S(a, b, c, d)$ is subnormal.*

Recall that the Bergman shift $B_+ \equiv shift\left(\sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \cdots\right)$ is subnormal with Berger measure $\mu = ds$. We let $B_+^\circ := B_+ \circ B_+ \equiv shift\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots\right)$. Then by (2.2), B_+° is subnormal. From ([11, Theorem 2.8], [14]), we knew that the Aluthge transform $\widetilde{B}_+$ of the Bergman shift B_+ and the mean transform $\widehat{B_+^\circ}$ of B_+° are both subnormal. Thus, we can ask whether the harmonic transform $\widetilde{B_+}^H$ (resp. the quadratic transform $\widetilde{B_+}^Q$ of B_+ is subnormal. We also ask whether $\widetilde{B_+^\circ}^H$ (resp. $\widetilde{B_+^\circ}^Q$ of B_+° is subnormal. We recall the following result.

Lemma 2.4. ([14])

(i) *The Aluthge transform $\widetilde{B_+^\circ}$ of B_+° is subnormal.*

(ii) *The Mean transform* $\widehat{B_+^\circ}$ *of* B_+° *is subnormal.*

Then we have:

THEOREM 2.5. *The quadratic transform* $\widetilde{B_+}^Q$ *of* B_+ *is subnormal.*

Proof. We let $B_+ = shift\left(\sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \cdots\right) := shift(\alpha_0, \alpha_1, \cdots)$ and $\widetilde{B_+}^Q := shift\left(\widetilde{\alpha}_0^Q, \widetilde{\alpha}_1^Q, \cdots\right)$. Then for $n \geq 0$, we have

$$\widetilde{\alpha}_n^Q = \sqrt{\frac{\alpha_n^2+\alpha_{n+1}^2}{2}} = \sqrt{\frac{\frac{n+1}{n+2}+\frac{n+2}{n+3}}{2}} = \left(\sqrt{\frac{\sqrt{2}n+2\sqrt{2}-1}{n+2}}\right)\left(\sqrt{\frac{\sqrt{2}n+2\sqrt{2}+1}{n+3}}\right)$$

and

$$\widetilde{B_+}^Q = S(\sqrt{2}, 2\sqrt{2}-1, 1, 2) \circ S(\sqrt{2}, 2\sqrt{2}+1, 1, 2).$$

By Lemma 2.3, we can see that $S(\sqrt{2}, 2\sqrt{2}-1, 1, 2)$ and $S(\sqrt{2}, 2\sqrt{2}+1, 1, 2)$ are both subnormal. Thus, by (2.2), we have that the quadratic transform $\widetilde{B_+}^Q$ of B_+ is subnormal, as desired. □

THEOREM 2.6. *The harmonic transform* $\widetilde{B_+^\circ}^H$ *of* B_+° *is subnormal.*

Proof. We let $\widetilde{B_+}^H := shift\left(\widetilde{\alpha}_0^H, \widetilde{\alpha}_1^H, \cdots\right)$. Then for $n \geq 0$, we can see that

$$\begin{aligned}
\widehat{\alpha}_n &= \frac{2\alpha_n\alpha_{n+1}}{\alpha_n+\alpha_{n+1}} = (\alpha_n\alpha_{n+1})\left(\frac{2}{\alpha_n+\alpha_{n+1}}\right) \\
&= \left(\frac{n+1}{n+3}\right)\left(\frac{2(n+2)(n+3)}{(n+1)(n+3)+(n+2)^2}\right) \\
&= \left(\frac{n+1}{n+3}\right)\left(\frac{2(n+2)^2+2(n+2)}{2(n+2)^2-1}\right) \\
&= \left(\frac{n+1}{n+3}\right)\left(\frac{2(n+2)+2}{\sqrt{2}(n+2)+1}\right)\left(\frac{n+2}{\sqrt{2}(n+2)-1}\right) \\
&= \left(\frac{n+1}{n+3}\right)\left(\frac{2n+6}{\sqrt{2}n+2\sqrt{2}+1}\right)\left(\frac{n+2}{\sqrt{2}n+2\sqrt{2}-1}\right) \\
&= \left(\sqrt{\frac{n+1}{n+3}}\right)\left(\sqrt{\frac{n+1}{n+3}}\right)\left(\sqrt{\frac{2n+6}{\sqrt{2}n+2\sqrt{2}+1}}\right)\left(\sqrt{\frac{2n+6}{\sqrt{2}n+2\sqrt{2}+1}}\right) \\
&\quad \times \left(\sqrt{\frac{n+2}{\sqrt{2}n+2\sqrt{2}-1}}\right)\left(\sqrt{\frac{n+2}{\sqrt{2}n+2\sqrt{2}-1}}\right).
\end{aligned}$$

Thus, we have that

$$\widetilde{B_+}^H = S(1,1,1,3)\circ S(1,1,1,3)\circ S(2,6,\sqrt{2},2\sqrt{2}+1)$$
$$\circ S(2,6,\sqrt{2},2\sqrt{2}+1)\circ S(1,2,\sqrt{2},2\sqrt{2}-1)\circ S(1,2,\sqrt{2},2\sqrt{2}-1).$$

Since $S(1,1,1,3)$, $S(2,6,\sqrt{2},2\sqrt{2}+1)$ and $S(1,2,\sqrt{2},2\sqrt{2}-1)$ are all subnormal, it follows from (2.2) that the harmonic transform $\widetilde{B_+^\circ}^H$ of B_+° is subnormal. $\square$

Next, we study that the k-hyponormality is not invariant under those transforms for $k \geq 1$. For them, we first recall that W_x is k-hyponormal if and only if for all $k \geq 1$ and $n \geq 0$, the Hankel matrix

$$H(k;n)\,(W_x) = \left(\gamma_{n+i+j-2}\,(W_x)\right)_{i,j=1}^{k+1} \geq 0 \text{ (Lemma 2.1)}, \tag{2.3}$$

For a weighted shift $W_\alpha \equiv shift(\alpha_0,\alpha_1,\alpha_2,\cdots)$, we also recall that
(2.4)

$$\widetilde{W}_\alpha = shift(\sqrt{\alpha_0\alpha_1},\sqrt{\alpha_1\alpha_2},\cdots);\ \widehat{W}_\alpha = shift\left(\tfrac{\alpha_0+\alpha_1}{2},\tfrac{\alpha_1+\alpha_2}{2},\cdots\right);$$

$$\widetilde{W}_\alpha^H := shift\left(\tfrac{2\alpha_0\alpha_1}{\alpha_0+\alpha_1},\tfrac{2\alpha_1\alpha_2}{\alpha_1+\alpha_2},\cdots\right);\ \widetilde{W}_\alpha^Q := shift\left(\sqrt{\tfrac{\alpha_0^2+\alpha_1^2}{2}},\sqrt{\tfrac{\alpha_1^2+\alpha_2^2}{2}},\cdots\right).$$

Next, we show that if W_α is subnormal with Berger measure $\mu = a\delta_p + (1-a)\,\delta_q$ for some $p,q>0$ $(p\neq q)$, then $\widetilde{W}_\alpha^H$ $\left(\text{resp. } \widetilde{W}_\alpha^Q\right)$ is never subnormal. For this, we recall *recursively generated weighted shifts* [5], [6]. We briefly recall some key facts about these shifts, specifically the case when there are two coefficients of recursion. In [18], J. Stampfli proved that given three positive numbers $\sqrt{a} < \sqrt{b} < \sqrt{c}$, it is always possible to find a subnormal weighted shift, denoted $W_{(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$, whose first three weights are $\sqrt{a},\sqrt{b}$ and $\sqrt{c}$. In this case, the coefficients of recursion (cf. [5, Example 3.12], [6, Section 3], [4, Section 1, p. 81]) are given by

$$\varphi_0 = -\frac{ab(c-b)}{b-a} \text{ and } \varphi_1 = \frac{b(c-a)}{b-a}, \tag{2.5}$$

the atoms t_0 and t_1 are the roots of the equation

$$t^2 - (\varphi_0 + \varphi_1 t) = 0, \tag{2.6}$$

and the densities ρ_0 and ρ_1 uniquely solve the 2×2 system of equations

$$\left\{\begin{array}{rcl} \rho_0+\rho_1 & = & 1 \\ \rho_0 t_0 + \rho_1 t_1 & = & \alpha_0^2. \end{array}\right. \tag{2.7}$$

Thus, we get

$$\mu = \rho_0 \delta_{t_0} + \rho_1 \delta_{t_1} \tag{2.8}$$

which is the Berger measure of $W_{(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$. We consider a recursively generated weighted shift $W_\alpha \equiv W_{\left(1,\sqrt{2},\sqrt{3}\right)^\wedge}$. Then by (2.8), we have that the Berger measure of $W_\alpha \equiv W_{\left(1,\sqrt{2},\sqrt{3}\right)^\wedge}$ is

$$\mu := \frac{1}{4}\left(\left(2+\sqrt{2}\right)\delta_{\left(2-\sqrt{2}\right)} + \left(2-\sqrt{2}\right)\delta_{\left(2+\sqrt{2}\right)}\right).$$

We let $p := \frac{2+\sqrt{2}}{4}$ and $q := \frac{2-\sqrt{2}}{4}$. Then the Aluthge transform $\widetilde{W}_\alpha$ of $W_\alpha \equiv W_{\left(1,\sqrt{2},\sqrt{3}\right)^\wedge}$ is

$$\widetilde{W}_\alpha \equiv shift\left(\sqrt{2}, \sqrt{6}, \sqrt{10}, \cdots\right)$$

and the mean transform $\widehat{W}_\alpha$ of $W_{(1,2,3)^\wedge}$ is

$$\widehat{W}_\alpha \equiv shift\left(\frac{1+\sqrt{2}}{2}, \frac{\sqrt{2}+\sqrt{3}}{2}, \frac{\sqrt{3}+\sqrt{\frac{10}{3}}}{2}, \cdots\right).$$

Also the harmonic transform $\widetilde{W}_\alpha^H$ of $W_{(1,2,3)^\wedge}$ is

$$\widetilde{W}_\alpha^H \equiv shift\left(\sqrt{\frac{8}{3+2\sqrt{2}}}, \sqrt{\frac{24}{5+2\sqrt{6}}}, \sqrt{\frac{120}{19+6\sqrt{10}}}, \cdots\right)$$

and the quadratic transform $\widetilde{W}_\alpha^Q$ of $W_{(1,2,3)^\wedge}$ is

$$\widetilde{W}_\alpha^Q \equiv shift\left(\sqrt{\frac{3}{2}}, \sqrt{\frac{5}{2}}, \sqrt{\frac{19}{6}}, \cdots\right).$$

Note that

$$H(2;0)\left(\widetilde{W}_\alpha\right),\ H(2;0)\left(\widehat{W}_\alpha\right),\ H(2;0)\left(\widetilde{W}_\alpha^H\right),\ H(2;0)\left(\widetilde{W}_\alpha^Q\right) < 0.$$

Thus, by Lemma 2.1,$\widetilde{W}_\alpha$, $\widehat{W}_\alpha$, $\widetilde{W}_\alpha^H$ and $\widetilde{W}_\alpha^Q$ are all not 2-hyponormal, so that $\widetilde{W}_\alpha$, $\widehat{W}_\alpha$, $\widetilde{W}_\alpha^H$ and $\widetilde{W}_\alpha^Q$ are not subnormal.

If we summarize these results just given above, we have:

Example 2.7. We consider a recursively generated weighted shift $W_\alpha \equiv W_{\left(1,\sqrt{2},\sqrt{3}\right)^\wedge}$. Then W_α is subnormal with Berger measure $\mu = a\delta_p + (1-a)\,\delta_q$ for some $p := \frac{2+\sqrt{2}}{4}, q := \frac{2-\sqrt{2}}{4} > 0$ $(p \neq q)$. But $\widetilde{W}_\alpha$, $\widehat{W}_\alpha$, $\widetilde{W}_\alpha^H$ and $\widetilde{W}_\alpha^Q$ are never subnormal.

By Lemmas 2.1, 2.4, Theorems 2.5, 2.6, the Nested Determinants Test (or Choleski's Algorithm) and direct computations, we have more:

Example 2.8. For $0 < x \leq \frac{3}{5}$, let $W_x \equiv shift\left(x, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots\right)$. Then we have

(i) W_x is not 2-hyponormal;
(ii) $\widetilde{W}_x$ is 2-hyponormal if and only if $0 < x \leq \frac{5}{9} \simeq 0.5556$;
(iii) $\widehat{W}_x$ is 2-hyponormal if and only if

$$0 < x \leq \tfrac{-82300568+1581\sqrt{9824630305}}{123450852} \simeq 0.5543;$$

(iv) $\widetilde{W}_x^H$ is not 2-hyponormal.

Example 2.9. For $W_\alpha \equiv shift\left(\alpha_0, \alpha_1, \alpha_2, \cdots\right)$, we let $\alpha_0 = \alpha_1 := \frac{1}{2}$ and consider the following relation:

$$\frac{\alpha_n + \alpha_{n+1}}{2} = \frac{n+1}{n+2} \text{ for all } n \geq 0.$$

We also let $\widehat{W}_\alpha := B_+^\circ \equiv shift\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots\right)$. Then we have

(i) $W_\alpha \equiv shift\left(\frac{1}{2}, \frac{1}{2}, \frac{5}{6}, \frac{2}{3}, \frac{14}{15}, \frac{11}{15}, \frac{103}{105}, \cdots\right)$ is not hyponormal;
(ii) $\widetilde{W}_\alpha$ is not 2-hyponormal;
(iii) $\widehat{W}_\alpha$ is subnormal;
(iv) $\widetilde{W}_\alpha^H$ is not hyponormal.

Example 2.10. For $\frac{1}{3} < x < \frac{2}{3}$, let $W_x \equiv shift\left(\frac{1}{3}, x, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots\right)$. Then we have

(i) W_x is 2-hyponormal if and only if $0 < x \leq \sqrt{\frac{2511}{7456}} \simeq 0.5803$;
(ii) $\widetilde{W}_x$ is not 2-hyponormal;
(iii) $\widehat{W}_x$ is 2-hyponormal if and only if

$$0 < x \leq \tfrac{-82300568+1581\sqrt{9824630305}}{123450852} \simeq 0.6027;$$

(iv) $\widetilde{W}_x^H$ is 2-hyponormal if and only if $0 < x \leq 0.4380$.

Example 2.11. For $0 < x \leq \frac{1}{2}$, let $W_x \equiv shift\left(\sqrt{x}, \sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \cdots\right)$. Then we have

(i) W_x is 2-hyponormal if and only if $0 < x \leq \frac{1}{3} \simeq 0.3333$;
(ii) $\widetilde{W}_x$ is 2-hyponormal if and only if

$$0 < x \leq \tfrac{3\left(77723+19164\sqrt{5}-9980\sqrt{6}-10820\sqrt{30}\right)}{303601} \simeq 0.3643;$$

(iii) $\widetilde{W}_x^Q$ is 2-hyponormal if and only if

$$0 < x \le \frac{-1883283+476\sqrt{55870729}}{3766566} \simeq 0.4446.$$

In ([13], [14]), we show that for a subnormal weighted shift W_α with finite Berger measure $a\delta_p + (1-a)\,\delta_q$ $(0 < a < 1,\ p < q)$, the Aluthge transform $\widetilde{W_\alpha}$ $\left(\text{resp. mean transform } \widehat{W_\alpha}\right)$ of W_α is subnormal if and only if $p = 0$. Thus, it is natural to ask whether the harmonic transform $\widetilde{W}_\alpha^H$ $\left(\text{resp. the quadratic transform } \widetilde{W}_\alpha^Q\right)$ of W_α has the same property just mentioned above. For this, we assume that W_α is a contractive subnormal weighted shift with Berger measure $\mu = a\delta_p + (1-a)\,\delta_1$ $(0 \le p < 1)$. If $a = 0$, then $W_\alpha \equiv U_+$ is subnormal with $\widetilde{W}_\alpha^H \equiv \widetilde{W}_\alpha^Q \equiv U_+$. Thus, $\widetilde{W}_\alpha^H$ and $\widetilde{W}_\alpha^Q$ are subnormal. If $a = 1$, then $W_\alpha \equiv \sqrt{p}\cdot U_+$ is subnormal with $\widetilde{W}_\alpha^H \equiv \widetilde{W}_\alpha^Q \equiv \sqrt{p}\cdot U_+$. Thus, $\widetilde{W}_\alpha^H$ and $\widetilde{W}_\alpha^Q$ are also subnormal. If $p = 0$, then $W_\alpha \equiv shift\left(\sqrt{1-a}, 1, 1, \cdots\right)$ is subnormal with $\widetilde{W}_\alpha^H \equiv shift\left((1-a)^{\frac{1}{4}}, 1, 1, \cdots\right)$ $\left(\text{resp. } \widetilde{W}_\alpha^Q \equiv shift\left(\frac{2\sqrt{1-a}}{1+\sqrt{1-a}}, 1, 1, \cdots\right)\right)$. Hence, $\widetilde{W}_\alpha^H$ and $\widetilde{W}_\alpha^Q$ are subnormal. Therefore, we have that for $a = 0, 1$, if W_α is a contractive subnormal with Berger measure $\mu = a\delta_0 + (1-a)\,\delta_1$ $(0 \le p < 1)$, then $\widetilde{W}_\alpha^H$ and $\widetilde{W}_\alpha^Q$ are both subnormal, as desired. For the back implication, we assume that $\widetilde{W}_\alpha^H$ $\left(\text{resp. } \widetilde{W}_\alpha^Q\right)$ is subnormal. Then by Lemma 2.1, the Hankel matrix

$$H(k;n)\left(\widetilde{W}_\alpha^H\right) = \left(\gamma_{n+i+j-2}\left(\widetilde{W}_\alpha^H\right)\right)_{i,j=1}^{k+1} \ge 0 \text{ for all } k \ge 1 \text{ and } n \ge 0.$$

Note that

$$\gamma_i\left(W_\alpha\right) = ap^i + 1 - a \text{ for } i \ge 1,$$

(2.9)

$$\gamma_i\left(\widetilde{W}_\alpha^H\right) = \frac{2^i\left(\sqrt{ap^2+1-a}\right)\left(\sqrt{\frac{ap^3+1-a}{ap+1-a}}\right)\cdots}{\left(\sqrt{ap+1-a}+\sqrt{\frac{ap^2+1-a}{ap+1-a}}\right)^2\left(\sqrt{\frac{ap^2+1-a}{ap+1-a}}+\sqrt{\frac{ap^3+1-a}{ap^2+1-a}}\right)^2\cdots}$$
$$\times \frac{\left(\sqrt{\frac{(ap^{(i+1)}+1-a)}{ap^{(i-1)}+1-a}}\right)}{\left(\sqrt{\frac{ap^i+1-a}{ap^{(i-1)}+1-a}}+\sqrt{\frac{ap^{(i+1)}+1-a}{ap^i+1-a}}\right)^2}.$$

and

(2.10)

$$\gamma_i\left(\widetilde{W}_\alpha^Q\right) = \frac{\left(ap+1-a+\frac{ap^2+1-a}{ap+1-a}\right)\left(\frac{ap^2+1-a}{ap+1-a}+\frac{ap^3+1-a}{ap^2+1-a}\right)\cdots\left(\frac{ap^i+1-a}{ap^{(i-1)}+1-a}+\frac{ap^{(i+1)}+1-a}{ap^i+1-a}\right)}{2^i}.$$

Case 1: If $p = 0$ or $a = 0, 1$, then it is clear.
Case 2: For $0 < p < 1$, we note that

$$
\begin{aligned}
&H(3;1)\left(\widetilde{W}_{\alpha}^{\#}\right) \geq 0 \\
&\Longleftrightarrow \begin{pmatrix}
\gamma_1\left(\widetilde{W}_{\alpha}^{\#}\right) & \gamma_2\left(\widetilde{W}_{\alpha}^{\#}\right) & \gamma_3\left(\widetilde{W}_{\alpha}^{\#}\right) & \gamma_4\left(\widetilde{W}_{\alpha}^{\#}\right) \\
\gamma_2\left(\widetilde{W}_{\alpha}^{\#}\right) & \gamma_3\left(\widetilde{W}_{\alpha}^{\#}\right) & \gamma_4\left(\widetilde{W}_{\alpha}^{\#}\right) & \gamma_5\left(\widetilde{W}_{\alpha}^{\#}\right) \\
\gamma_3\left(\widetilde{W}_{\alpha}^{\#}\right) & \gamma_4\left(\widetilde{W}_{\alpha}^{\#}\right) & \gamma_5\left(\widetilde{W}_{\alpha}^{\#}\right) & \gamma_6\left(\widetilde{W}_{\alpha}^{\#}\right) \\
\gamma_4\left(\widetilde{W}_{\alpha}^{\#}\right) & \gamma_5\left(\widetilde{W}_{\alpha}^{\#}\right) & \gamma_6\left(\widetilde{W}_{\alpha}^{\#}\right) & \gamma_7\left(\widetilde{W}_{\alpha}^{\#}\right)
\end{pmatrix} \\
&\quad := \begin{pmatrix}
D\left(\widetilde{W}_{\alpha}^{\#}\right) & E\left(\widetilde{W}_{\alpha}^{\#}\right) \\
\left(E\left(\widetilde{W}_{\alpha}^{\#}\right)\right)^* & F\left(\widetilde{W}_{\alpha}^{\#}\right)
\end{pmatrix} =: P\left(H(3;1)\left(\widetilde{W}_{\alpha}^{\#}\right)\right) \geq 0,
\end{aligned}
$$

where

$$
\begin{aligned}
D\left(\widetilde{W}_{\alpha}^{\#}\right) &:= \begin{pmatrix}
\gamma_1\left(\widetilde{W}_{\alpha}^{\#}\right) & \gamma_2\left(\widetilde{W}_{\alpha}^{\#}\right) \\
\gamma_2\left(\widetilde{W}_{\alpha}^{\#}\right) & \gamma_3\left(\widetilde{W}_{\alpha}^{\#}\right)
\end{pmatrix}, \\
E\left(\widetilde{W}_{\alpha}^{\#}\right) &:= \begin{pmatrix}
\gamma_3\left(\widetilde{W}_{\alpha}^{\#}\right) & \gamma_4\left(\widetilde{W}_{\alpha}^{\#}\right) \\
\gamma_4\left(\widetilde{W}_{\alpha}^{\#}\right) & \gamma_5\left(\widetilde{W}_{\alpha}^{\#}\right)
\end{pmatrix}, \\
F\left(\widetilde{W}_{\alpha}^{H}\right) &:= \begin{pmatrix}
\gamma_5\left(\widetilde{W}_{\alpha}^{\#}\right) & \gamma_6\left(\widetilde{W}_{\alpha}^{\#}\right) \\
\gamma_6\left(\widetilde{W}_{\alpha}^{\#}\right) & \gamma_7\left(\widetilde{W}_{\alpha}^{\#}\right)
\end{pmatrix} \text{ and } \widetilde{W}_{\alpha}^{\#} := \widetilde{W}_{\alpha}^{H} \text{ or } \widetilde{W}_{\alpha}^{Q}.
\end{aligned}
$$

We now apply Lemma 2.2 to $P\left(H(3;1)\left(\widetilde{W}_{\alpha}^{\#}\right)\right)$. A direct calculation shows that $F\left(\widetilde{W}_{\alpha}^{\#}\right)$ is invertible on $\{(a,p) : (a,p) \in (0,1) \times (0,1)\}$. Hence, we have

$$
\begin{aligned}
&P\left(H(3;1)\left(\widetilde{W}_{\alpha}^{\#}\right)\right) \geq 0 \\
&\Longleftrightarrow D\left(\widetilde{W}_{\alpha}^{\#}\right) - \left(W\left(\widetilde{W}_{\alpha}^{\#}\right)\right)^* F\left(\widetilde{W}_{\alpha}^{\#}\right) W\left(\widetilde{W}_{\alpha}^{\#}\right) \geq 0,
\end{aligned}
$$

where $W\left(\widetilde{W}_{\alpha}^{\#}\right) := F\left(\widetilde{W}_{\alpha}^{\#}\right)^{-1}\left(E\left(\widetilde{W}_{\alpha}^{\#}\right)\right)^*$.

A direct calculation shows that

$$
\begin{aligned}
&D\left(\widetilde{W}_{\alpha}^{\#}\right) - \left(W\left(\widetilde{W}_{\alpha}^{\#}\right)\right)^* F\left(\widetilde{W}_{\alpha}^{\#}\right) W\left(\widetilde{W}_{\alpha}^{\#}\right) \geq 0 \\
&\Longleftrightarrow Q\left(\widetilde{W}_{\alpha}^{\#}\right) := \begin{pmatrix}
q_{11}\left(\widetilde{W}_{\alpha}^{\#}\right) & q_{12}\left(\widetilde{W}_{\alpha}^{\#}\right) \\
q_{21}\left(\widetilde{W}_{\alpha}^{\#}\right) & q_{22}\left(\widetilde{W}_{\alpha}^{\#}\right)
\end{pmatrix} \geq 0.
\end{aligned}
$$

Note that
$Q\left(\widetilde{W}_\alpha^{\#}\right) \geq 0$ if and only if $q_{11}\left(\widetilde{W}_\alpha^{\#}\right) \geq 0,\ q_{11}\left(\widetilde{W}_\alpha^{\#}\right) \geq 0$ and $\det Q\left(\widetilde{W}_\alpha^{\#}\right) \geq 0.$

Using the software tool *Mathematica* [19], we can observe that

$$\det Q\left(\widetilde{W}_\alpha^{\#}\right) < 0 \text{ when } (a,p) \in (0,1) \times (0,1)\,.$$

Thus $\widetilde{W}_\alpha^{\#}$ is not 3-hyponormal, that is, $\widetilde{W}_\alpha^{\#}$ is not subnormal when $(a,p) \in (0,1)\times(0,1)$. Therefore, $\widetilde{W}_\alpha^{\#}$ is not subnormal when $0 < p < 1$, which is contradict to the assumption. Therefore, by **Case 1** and **Case 2** , the mean transform $\widetilde{W}_\alpha^{\#}$ is subnormal only if $p = 0$.

If we summarize these results just given above, we have:

THEOREM 2.12. *Let W_α be a contractive subnormal weighted shift with Berger measure $\mu = a\delta_p + (1-a)\,\delta_1$ $(0 \leq p < 1)$. Then we have that*

(i) *the harmonic transform $\widetilde{W}_\alpha^{H}$ of W_α is subnormal if and only if $p = 0$ or $a = 0, 1$.*

(ii) *the quadratic transform $\widetilde{W}_\alpha^{Q}$ of W_α is subnormal if and only if $p = 0$ or $a = 0, 1$.*

COROLLARY 2.13. *Let $W_\alpha \equiv shift(\alpha_0, \alpha_1, \cdots)$ be a subnormal with Berger measure*

$$\mu = a\delta_p + (1-a)\,\delta_q \ \ (0 < a < 1,\, p < q)\,.$$

(i) *the harmonic transform $\widetilde{W}_\alpha^{H}$ of W_α is subnormal if and only if $p = 0$.*

(ii) *the quadratic transform $\widetilde{W}_\alpha^{Q}$ of W_α is subnormal if and only if $p = 0$.*

Proof. $(\Longleftarrow)$ Suppose that $p = 0$. Then a direct calculation shows that W_α is

$$shift\left(\sqrt{(1-a)\,q}, \sqrt{q}, \sqrt{q}, \cdots\right),$$

so that we have

$$\widetilde{W}_\alpha^{H} \equiv shift\left(\frac{2\sqrt{(1-a)\,q}}{1+\sqrt{(1-a)}}, \sqrt{q}, \sqrt{q}, \cdots\right)$$

and

$$\widetilde{W}_\alpha^{Q} \equiv shift\left(\sqrt{\frac{(2-a)\,q}{2}}, \sqrt{q}, \sqrt{q}, \cdots\right).$$

Thus, $\widetilde{W}_\alpha^H$ and $\widetilde{W}_\alpha^H$ are subnormal with Berger measures

$$\left(1-\left(\frac{2\sqrt{(1-a)}}{1+\sqrt{(1-a)}}\right)^2\right)\delta_0+\left(\frac{2\sqrt{(1-a)}}{1+\sqrt{(1-a)}}\right)^2\delta_q \text{ and} \tag{2.11}$$

$$\left(1-\frac{(2-a)}{2}\right)\delta_0+\left(\frac{(2-a)}{2}\right)\delta_q, \text{ respectively.}$$

($\Longrightarrow$) We note that if W_α is a subnormal with $a\delta_q+(1-a)\,\delta_q$, then $\left(\frac{1}{\sqrt{q}}\right)W_\alpha$ is a subnormal with Berger measure $a\delta_{\frac{p}{q}}+(1-a)\,\delta_1$. Suppose that $\widetilde{W}_\alpha^H$ $\left(\text{resp. } \widetilde{W}_\alpha^Q\right)$ is subnormal. Then $\left(\frac{1}{\sqrt{q}}\right)\widetilde{W}_\alpha^H$ $\left(\text{resp. } \left(\frac{1}{\sqrt{q}}\right)\widetilde{W}_\alpha^Q\right)$ is also subnormal. Note that $\left(\frac{1}{\sqrt{q}}\right)\widetilde{W}_\alpha^H$ $\left(\text{resp. } \left(\frac{1}{\sqrt{q}}\right)\widetilde{W}_\alpha^Q\right)$ is the harmonic transform (resp. quadratic transform) of $\left(\frac{1}{\sqrt{q}}\right)W_\alpha$. Thus, by Theorem 2.12, we have $p=0$, as desired. □

REMARK 2.14. By Example 2.7, Theorem 2.12 and Corollary 2.13, it is nature to conjecture: Let W_α be a subnormal with finite atomic Berger measure μ. Then the harmonic transform $\widetilde{W}_\alpha^H$ (resp. the quadratic transform $\widetilde{W}_\alpha^Q$) of W_α is subnormal if and only if $\mu=a\delta_0+(1-a)\,\delta_p$ $(0<a<1)$.

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